ON SMALL HARMONIC OSCILLATIONS OF A CYLINDRICAL SHELL ALONG THE AXIS OF WHICH AN IDEAL GAS FLOWS WITH SUPERSONIC VELOCITY

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A circular cylindrical shell is considered to have a flat bottom at one end where a system of uniformly distributed supersonic sources is located. The other end of the shell is open and through it flows a uniform supersonic stream of an ideal gas originating at the bottom. On the assumption that the shell performs small harmonic oscillations in a certain plane, the dynamic interaction between the gas and the shell walls is investigated. The gas compressibility leads to the appearance of nonstationary forces whose role in the general scheme depends upon the Strouhal number; in other words, the principal vector of the gas-dynamic forces manifests itself during the shell oscillation as displacement and rotation relative to the longitudinal axis.

1. Formulation of the problem. The following notation is introduced: S_1 = area of the shell bottom; S_2 = area of the side surface of the shell; R_0 = shell radius; h = length of the generator; Q = volume of gas inside the shell; c = displacement velocity of the gas; ρ_0 = mass density of the undisturbed gas, and

$$\mu^{\cdot} = \left|\frac{d\mu}{dt}\right| = \rho_0 c S = \rho_0 c \pi R_0^2$$

the mass efflux per second.

The pressure in the surrounding medium is taken equal to the pressure in the gas stream.

For describing the motion of the gas we introduce an "associated" coordinate system $Oxyz$ with an origin at the center of the circle S_1, The Ox-axis parallel to S_2, the Oy-axis lying in the plane of the disturbed motion; and corresponding to this an absolute coordinate system

$O^*x^*y^*z^*$ which coincides with the $Oxyz$ system in the absence of any disturbance. Subsequently, by an undisturbed motion we mean either a state of rest or motion of the $O^*x^*y^*z^*$ coordinate system such that the inertia forces in its field are neglected when considering a disturbance in the gas. The motion of $Oxyz$ relative to $O^*x^*y^*z^*$ will be characterized by the velocity vector $\mathbf{u}$ of the point O and by an angular velocity ω, supposed to be a small quantity of the first order:

$$\mathbf{u} = u\mathbf{i}_2, \qquad \omega = \omega\mathbf{i}_3 \tag{1.1}$$

where $\mathbf{i}_2$ and $\mathbf{i}_3$ are unit vectors in the $O^*y^*z^*$ coordinate system.

Quantities of the second and higher orders are neglected. The shell motion in the O^*x^*-direction due to the disturbance is of minor interest and so it will be assumed that $u_x \equiv 0$.

In order to construct the velocity field the following assumptions are made:

(1) the components of the velocity vector of the disturbance at any point in the region Q are small compared with c;

(2) the gas flow is a potential flow.

Consider the absolute flow of the gas in the associated coordinate system, i.e. fix attention on the $Oxyz$ system at the instant when it coincides with the $O^*x^*y^*z^*$ system during the disturbed motion. Then the linearized equation for the velocity potential Φ may be written in the following well-known form

$$-(M^2-1)\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{2M}{a}\frac{\partial^2\Phi}{\partial x\partial t} - \frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \tag{1.2}$$

where a is the velocity of sound, $M = c/a > 1$ is a certain constant, and the time derivatives are calculated in the $O^*x^*y^*z^*$ coordinate system. For the formulation of the boundary conditions we assume that the gas particles close to S_1 move with S_1 and rotate about Oz-axis as a solid body during the displacement and rotation of the bottom S_1. Normal components of gas velocity must be created on the surface S_2. By virtue of the supersonic character of the flow, this cylindrical surface may be considered as semi-infinite, so that any additional condition at the open end is unnecessary.

Thus, at the instant when $Oxyz$ coincides with $O^*x^*y^*z^*$,

$$\frac{\partial\Phi}{\partial s} = (\mathbf{u}, \mathbf{s}) + (\mathbf{R}\times\mathbf{s}, \boldsymbol{\omega}), \qquad \frac{\partial\Phi}{\partial\nu} = (\mathbf{R}\times\boldsymbol{\nu}, \boldsymbol{\omega}) \qquad \text{on } S_1$$

$$\frac{\partial\Phi}{\partial\nu} = (\mathbf{u}, \boldsymbol{\nu}) + (\mathbf{R}\times\boldsymbol{\nu}, \boldsymbol{\omega}) \text{ on } S_2 \tag{1.3}$$

Here ν is a unit vector normal to Q, and $\mathbf{s}$ is a unit vector in any direction of the S_1 plane. For harmonic oscillations, one may write

$$u(t) = u_0 e^{i\sigma t}, \qquad \omega(t) = \omega_0 e^{i\sigma t} \tag{1.4}$$

2. Determination of the velocity potential of the disturbed gas flow. We seek the velocity potential in the form of a sum of terms corresponding to an incompressible fluid flow inside an infinite cylinder S_2, and an infinite series expressing the effect of a wave system arising during the disturbed motion:

$$\Phi = yu + xy\omega + \sum_{n=1}^{\infty} \psi_n \frac{\partial r_n(x, t)}{\partial t} \tag{2.1}$$

where y and xy are harmonic functions satisfying the boundary conditions (1.3) and S_2, and where $\psi_n (n = 1, 2, \ldots)$ are eigen functions of the Helmholtz equation

$$\frac{\partial^2 \psi_n}{\partial y^2} + \frac{\partial^2 \psi_n}{\partial z^2} + k_n^2 \psi_n = 0 \tag{2.2}$$

In the region S representing a cross-section of Q, the relation

$$\partial \psi_n / \partial \nu = 0 \tag{2.3}$$

is determined from the boundary condition on the contour C which bounds S.

Because of the circular region under consideration, we introduce the following expression for the functions ψ_n in (2.1):

$$\psi_n = \frac{J_1(\xi_n R / R_0)}{J_1(\xi_n)} \cos\theta, \qquad k_n = \frac{\xi_n}{R_0} \qquad (n = 1, 2, \ldots) \tag{2.4}$$

where J_1 is the Bessel function of the first kind and first order, θ is the polar angle measured from Oy, R is the radius and ξ_n is a root of the equation

$$J_n'(\xi) = 0 \tag{2.5}$$

Substitution of expression (2.1) into equation (1.2), on the assumption that the series in (2.1) is differentiable term by term, gives the following differential equation and initial conditions for the function $r_n(x, t)$:

$$\left[a^2(M^2 - 1)\frac{\partial^2 r_n}{\partial x^2} + 2c\frac{\partial^2 r_n}{\partial x \partial t} + + \frac{\partial^2 r_n}{\partial t^2} + a^2 k_n^2 r_n\right] \psi_n + y(u^{\cdot} + 2c\omega + x\omega^{\cdot}) = 0 \tag{2.6}$$

$$r_n(0, t) = 0, \qquad \sum_{n=1}^{\infty} \frac{\partial^2 r_n(x, t)}{\partial x \partial t} = -2y\omega \qquad \text{for } x = 0 \tag{2.7}$$

after recalling condition (2.1) and neglecting an arbitrary function of the coordinates.

By use of the substitution $r_n(x, t) = \zeta_n(x)e^{i\sigma t}$ and expansion of the function $y = R\cos\theta$ in a generalized Fourier series of ψ_n functions,

$$y = \sum_{n=1}^{\infty} \frac{D_n}{N_n^2}\psi_n \qquad \left(D_n = \frac{\pi R_0^3}{\xi_n^2}, \quad N_n^2 = \frac{\pi R_0^2(\xi_n^2 - 1)}{2\xi_n^2}\right) \tag{2.8}$$

equations (2.6) and (2.7) may be brought into the following equivalent system of ordinary differential equations and initial conditions for the $\zeta_n(x)$ functions

$$a^2(M^2-1)\frac{d^2\zeta_n}{dx^2} + 2ic\sigma_n\frac{d\zeta_n}{dx} + (\sigma_n^2 - \sigma^2)\zeta_n = -\frac{D_n\sigma}{N_n^2}\left[iu_0 + \left(ix + \frac{2c}{\sigma}\omega_0\right)\right] \tag{2.9}$$

$$\zeta_n(0) = 0, \qquad \zeta_n'(0) = \frac{2iD_n\omega_0}{N_n^2\sigma} \qquad (n = 1, 2, \ldots) \tag{2.10}$$

where

$$\sigma_n^2 = k_n^2a^2 = \frac{\xi_n^2a^2}{R_0^2} \tag{2.11}$$

A solution of (2.9) which satisfies (2.10) may be written in the form

$$\zeta_n(x) = \frac{D_n}{N_n^2}\sum_{k=1}^{2}\alpha_k\zeta_{nk}(x) \tag{2.12}$$

where

$$\zeta_{n1}(x) = -\frac{\sigma^2}{\sigma_n^2 - \sigma^2}\{-1 + e^{-iM\delta\sigma x}[\cos\delta\sqrt{\beta_n^2 + \sigma^2}\,x + \frac{iM\sigma}{\sqrt{\beta_n^2+\sigma^2}}\sin\delta\sqrt{\beta_n^2 + \sigma^2}\,x]\} \tag{2.13}$$

$$\zeta_{n2}(x) = \frac{\sigma}{\sigma_n^2 - \sigma^2}\left(\sigma x - \frac{2ic\sigma_n^2}{\sigma_n^2 - \sigma^2}\right) + \frac{ae^{-iM\delta\sigma x}}{(\sigma_n^2-\sigma^2)^2}\Big\{2iM\sigma_n^2\sigma\cos\delta\sqrt{\beta_n^2+\sigma^2}\,x - \frac{(M^2-1)(\sigma^4 + 2\sigma_n^4 - \sigma^2\sigma_n^2) + 2\sigma^2\sigma_n^2}{\sqrt{\beta_n^2+\sigma^2}}\sin\delta\sqrt{\beta_n^2+\sigma^2}\,x\Big\}$$

$$\alpha_1 = u_0, \qquad \alpha_2 = \omega_0, \qquad \delta = \frac{1}{a(M^2-1)}, \qquad \beta_n^2 = \sigma_n^2(M^2-1)$$

3. Calculation of the principal vector and principal moment of the system of gas-dynamic forces. The principal vector $\mathbf{P}$ and the principal moment $\mathbf{M}_0$ of the system of gas-dynamic forces relative to the point O are most simply obtained by employing a function characteristic of the disturbed pressure and by taking into account an additional change in motion and kinetic moment resulting from efflux of gas from the sources on the surface S_1:

$$\mathbf{P} = \iint\limits_{S_1+S_2} \delta p \boldsymbol{\nu} dS + \mathbf{P}^{(r)}, \qquad \mathbf{M}_0 = \iint\limits_{S_1+S_2} \delta p (\mathbf{R} \times \boldsymbol{\nu})\, dS + \mathbf{M}_0^{(r)} \tag{3.1}$$

where

$$\mathbf{P}^{(r)} = -\,\mathbf{c}\mu^{\cdot}, \qquad \mathbf{M}_0^{(r)} = c^2 \iint\limits_{S_1} (\mathbf{R} \times \boldsymbol{\nu})\, \delta\rho dS \tag{3.2}$$

and $\mathbf{R}$ is the radius vector of a point on the surfaces $S_1 + S_2$. The variations of pressure δp and of density $\delta\rho$ may be expressed in the associated coordinates. The Lagrange-Cauchy integral in the movable $Oxyz$ system has the form

$$\frac{\partial \Phi}{\partial t} + \frac{v_a^{\,2}}{2} - \mathbf{v}_a \cdot \mathbf{v}_l + \int \frac{dp}{\rho} = \chi(t)$$

where $\mathbf{v}_a$ is the absolute velocity of the gas particles, $\mathbf{v}_l$ the transfer velocity, and $\chi(t)$ an arbitrary function of time.

In the case under consideration $\mathbf{v}_a = \mathbf{c} + \nabla \Phi$, $\mathbf{v}_l = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{R}$, ; therefore the expression for δp takes on the following form after neglecting the nonessential functions of time

$$\delta p = -\rho_0 \left(\frac{\partial \Phi}{\partial t} + c \frac{\partial \Phi}{\partial x} + cy\omega \right) \tag{3.3}$$

where ρ_0 is the mass density of the undisturbed gas. The expression for the variation $\delta\rho$ in density to the same degree of accuracy has the form

$$\delta\rho = \frac{\delta p}{a^2} = -\frac{\rho_0}{a^2} \left(\frac{\partial \Phi}{\partial t} + c \frac{\partial \Phi}{\partial x} + cy\omega \right) \tag{3.4}$$

By virtue of the boundary condition (1.3) on the surface S_1

$$\delta\rho = -\frac{\rho_0}{a^2} \frac{\partial \Phi}{\partial t}$$

By substitution of δp from (3.3) into formulas (3.1) and integrating over the surface $S_1 + S_2$, in the second case by parts, we get

$$\mathbf{P} = -\rho_0 \left[\iint\limits_{S_1+S_2} \frac{\partial \Phi}{\partial t} \boldsymbol{\nu} dS + c \iint\limits_{S_2} \frac{\partial \Phi}{\partial x} \boldsymbol{\nu} dS + hc\omega \oint\limits_{c} y \boldsymbol{\nu} ds \right] - \mathbf{c}\mu^{\cdot} \tag{3.5}$$

$$\mathbf{M}_0 = -\rho_0 \left[\iint\limits_{S_1+S_2} \frac{\partial \Phi}{\partial t} (\mathbf{R} \times \boldsymbol{\nu})\, dS + c \iint\limits_{S_2} \frac{\partial \Phi}{\partial x} (\mathbf{R} \times \boldsymbol{\nu})\, dS - \right.$$
$$\left. - \iint\limits_{S_2} \Phi (\mathbf{c} \times \boldsymbol{\nu})\, dS + c\omega \iint\limits_{S_2} y (\mathbf{R} \times \boldsymbol{\nu})\, dS + M^2 \iint\limits_{S_1} \frac{\partial \Phi}{\partial t} (\mathbf{R} \times \boldsymbol{\nu})\, dS \right]$$

A further substitution of Φ from (2.1) into (3.5) yields

$$\mathbf{P} = -\frac{d\mathbf{K}}{dt} - \mathbf{c}\mu^{\cdot} - 2\mu^{\cdot} (\boldsymbol{\omega} \times \mathbf{R}_0) - \rho_0 \sum_{n=1}^{\infty} \iint\limits_{S_2} \frac{\partial}{\partial t} \left(\frac{\partial r_n}{\partial t} + c \frac{\partial r_n}{\partial x} \right) \psi_n \boldsymbol{\nu} dS$$

$$\mathbf{M}_0 = -\frac{d\mathbf{L}_0}{dt} + \mu^{\cdot} (\omega \times \mathbf{R}_0) \times \mathbf{R}_0 - \rho_0 \sum_{n=1}^{\infty} \iint\limits_{S_2} \frac{\partial}{\partial t} \left(\frac{\partial r_n}{\partial t} + c \frac{\partial r_n}{\partial x} \right) (\mathbf{R} \times \boldsymbol{\nu})\, \psi_n dS \tag{3.6}$$

Here

$$\mathbf{K} = \mu(\mathbf{u} + \omega \times \mathbf{R}_c) = \mu\left(u + \frac{h}{2}\,\omega\right)\mathbf{i}_2$$

$$\mathbf{L}_0 = \mu\left[\mathbf{u} \times \mathbf{R}_c + \frac{2}{3}(\omega \times \mathbf{R}_c) \times \mathbf{R}_c\right] + \delta\mathbf{L}_0 =$$

$$= \mu\left[\frac{h}{2}\,u + \frac{h^2}{3}\,\omega + (1 + M^2)\frac{R_0^2 u}{h}\right]\mathbf{i}_3 \qquad (3.7)$$

where μ is the mass of gas filling the volume Q, $\mathbf{R}_0$ is the radius vector to the center of the exit section, $\mathbf{R}_c$ is the radius vector to the center of inertia of the gaseous mass Q

$$\mu = \pi R_0^2 h \rho_0, \qquad \mathbf{R}_0 = h\mathbf{i}_1, \qquad \mathbf{R}_c = \frac{h}{2}\,\mathbf{i}_1 \qquad (3.8)$$

The following formulas were used in obtaining expressions (3.6)

$$\sum_{n=1}^{\infty} \frac{1}{\xi_n^2 - 1} = \frac{1}{2}, \qquad \sum_{n=1}^{\infty} \frac{1}{\xi_n^2(\xi_n^2 - 2)} = \frac{1}{8} \qquad (3.9)$$

We introduce the operators

$$\Omega_{n1}(\zeta_{nk}) = \left[a^2(M^2 - 1)\frac{d}{dx} + ic\sigma\right]\zeta_{nk}(x)\Big|_{x=0}^{x=h} + \sigma_n^2\int_0^h \zeta_{nk}(x)\,dx$$

$$\Omega_{n2}(\zeta_{nk}) = h\left[a^2(M^2 - 1)\frac{d}{dh} + ic\sigma\right]\zeta_{nk}(h) + \sigma_n^2\int \zeta_{nk}(h)\,dh -$$

$$- \int_0^h \left\{\left[a^2(M^2 - 1)\frac{d}{dx} + ic\sigma\right]\zeta_{nk}(x)\,dx + \sigma_n^2\int \zeta_{nk}(x)\,dx\right\}dx \qquad (3.10)$$

and pass over to the scalar form of the equations (3.6). Substitution of the corresponding expression in (2.4) for the functions ψ_n, and recalling formulas (2.8), (2.11), (2.12) and (1.4), equation (3.6) gives after some squaring

$$P_y = 2\pi\rho_0 R_0^2 e^{i\sigma t}\sum_{n=1}^{\infty}\frac{1}{\xi_n^2 - 1}\sum_{k=1}^{2}\alpha_k\Omega_{n1}(\zeta_{nk}) + \frac{\mu c\omega_0 e^{i(\sigma t + 1/2\pi)}}{\sigma} \qquad (3.11)$$

$$M_{0z} = 2\pi\rho_0 R_0^2 e^{i\sigma t}\sum_{n=1}^{\infty}\frac{1}{\xi_n^2 - 1}\sum_{k=1}^{2}\alpha_k\Omega_{n2}(\zeta_{nk}) - \frac{\pi\rho_0 R_0^4\sigma}{4}(M^2 + 1)\,e^{i(\sigma t + 1/2\pi)}.$$

Returning to the functions $u(t)$ and $\omega(t)$ of equations (1.4) and introducing the notation

$$\vartheta = -\frac{\omega_0{}^{i(\sigma t + \pi/2)}}{\sigma} \qquad (3.12)$$

for small rotations of the $Oxyz$ system relative to the $O^*x^*y^*z^*$ system, formulas (3.11) may be given in the following form:

$$P_y = -[\mu_{11}(\sigma)u^{\cdot} + \mu_{12}(\sigma)\omega^{\cdot} + \lambda_{11}(\sigma)u + \lambda_{12}(\sigma)\omega + c\mu^{\cdot}\vartheta]$$
$$M_{0z} = -[\mu_{21}(\sigma)u^{\cdot} + \mu_{22}(\sigma)\omega^{\cdot} + \lambda_{21}(\sigma)u + \lambda_{22}(\sigma)\omega] \tag{3.13}$$

where

$$\mu_{jk} = \frac{2}{\sigma^2}\sum_{n=1}^{\infty} \mu_n \operatorname{Re}[\Omega_{nj}(\zeta_{nk})], \qquad \mu_n = \frac{\pi R_0^2 \rho_0}{\xi_n^2 - 1} \tag{3.14}$$
$$\lambda_{jk} = -\frac{2}{\sigma}\sum_{n=1}^{\infty} \mu_n J^m[\Omega_{nj}(\zeta_{nk})] \qquad (j,k=1,2),$$

The expressions for μ_{1k} and λ_{1k} $(k = 1, 2)$ obtained from (3.14), after substitution of $\zeta_{n1}(x)$ and $\zeta_{n2}(x)$ from (2.13) and squaring, reduce to

$$\mu_{11}(\sigma) = \sum_{n=1}^{\infty} \mu_n \Big\{[P_{11}^{(n)}(\sigma) - Q_{11}^{(n)}(\sigma)]S_1^{(n)}(\sigma) - [P_{11}^{(n)}(\sigma) + Q_{11}^{(n)}(\sigma)]S_2^{(n)}(\sigma) + \frac{2h}{R_0}P_{12}^{(n)}(\sigma)\Big\} \tag{3.15}$$

$$\lambda_{11}(\sigma) = \sigma\sum_{h=1}^{\infty} \mu_n \Big\{-[P_{11}^{(n)}(\sigma) - Q_{21}^{(n)}(\sigma)]C_1^{(n)}(\sigma) + [P_{11}^{(n)}(\sigma) + Q_{11}^{(n)}(\sigma)]C_2^{(n)}(\sigma) - 2Q_{11}^{(n)}(\sigma)\Big\}$$

$$\mu_{12}(\sigma) = \sum_{n=1}^{\infty} \mu_n R_0 \Big\{[P_{12}^{(n)}(\sigma) + Q_{12}^{(n)}(\sigma)]C_1^{(n)}(\sigma) - [P_{12}^{(n)}(\sigma) - Q_{12}^{(n)}(\sigma)]C_2^{(n)}(\sigma) - 2\Big[Q_{12}^{(n)}(\sigma) - \frac{h^2}{2R_0^2(1-\zeta_n^2)}\Big]\Big\}$$

$$\lambda_{12}(\sigma) = \sigma\sum_{n=1}^{\infty} \mu_n R_0 \Big\{[P_{12}^{(n)}(\sigma) + Q_{12}^{(n)}(\sigma)]S_1^{(n)}(\sigma) - [P_{12}^{(n)}(\sigma) - Q_{12}^{(n)}(\sigma)]S_2^{(n)}(\sigma)\Big\}$$

Here

$$S_{1,2}^{(n)}(\sigma) = \sin\delta\sigma_n(\zeta_n M \pm \sqrt{M^2 - 1 + \zeta_n^2})h$$
$$C_{1,2}^{(n)}(\sigma) = \cos\delta\sigma_n(\zeta_n M \pm \sqrt{M^2 - 1 + \zeta_n^2})h$$

$$P_{11}^{(n)}(\sigma) = -\frac{\zeta_n^2[2(M^2-1) + 1 + \zeta_n^2]}{\xi_n(1-\zeta_n^2)^2\sqrt{M^2 - 1 + \zeta_n^2}}, \quad Q_{11}^{(n)}(\sigma) = -\frac{M\zeta_n(1+\zeta_n^2)}{\xi_n(1-\zeta_n^2)^2} \tag{3.16}$$

$$P_{12}^{(n)}(\sigma) = -\frac{M[(M^2-1)(\zeta_n^4 + \zeta_n^2 + 2) + 2\zeta_n^4]}{\xi_n\zeta_n^2(1-\zeta_n^2)^3\sqrt{M^2 = 1 + \zeta_n^2}}$$

$$Q_{12}^{(n)}(\sigma) = \frac{(M^2-1)(\zeta_n^4 - \zeta_n^2 + 4) + 2(1+\zeta_n^2)}{\zeta_n^2(1-\zeta_n^2)^3} \qquad \left(\zeta_n = \frac{\sigma}{\sigma_n}\right)$$

It is evident from (3.15) and (3.16) that effects connected with the nonstationary process are nonessential, if the value of ζ_1^2 is negligibly small in comparison with unity, i.e. if the inequality

$$\left(\frac{\sigma R_0}{1.844\,a}\right)^2 \ll 1 \tag{3.17}$$

is satisfied, where $\xi_1 \approx 1.8442$.

Supposing, for example, that R_0/a 10^{-3} sec, we obtain in this case for the condition of applicability of the stationary hypothesis

$$\left(\frac{\sigma}{1.844}\right)^2 \ll 10^6$$

Noticeable deviations from this must be observed only for forced oscillations with frequencies of the order of hundreds of cycles per sec. In other words, a supersonic gas stream flowing through a cylindrical channel may be considered as "absolutely stiff" in the transverse direction for a wide range of Strouhal numbers. By passing to the limiting case of $\sigma \to 0$ in (3.14), expressions are obtained which indicate the range of applicability of the stationary hypothesis for the effect of the bottom S_1.

Translated by E.Z.S.